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Remarks on the geometric interpretation of superconductive flux quantisation

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Abstract. A complete discussion is presented of the interpretation of superconductive flux quantisation in terms of holonomy groups. The contrast between this effect and the flux quantisation effect expressed by the Dirac monopole charge condition is emphasised.

1. Introduction

In view of the central position which the concept of 'topological charge' has come to occupy in the theory of vortices, monopoles, and instantons (Bais 1982), it is natural to consider the question of whether the magnetic flux quantisation condition of superconductivity theory (Rose-Innes and Rhoderick 1978) has a similar interpretation. In fact, two independent ways of obtaining such an interpretation are possible within the standard formalism.

Consider an annular specimen of superconductive material immersed in a magnetic field. The field will penetrate the core, but (as a result of the Meissner effect) not the specimen itself. Since the field vanishes everywhere on the boundary of the core, the core region can be 'compactified' to S^2 (the 2-sphere), in just the same way as boundary conditions are used to compactify $\mathbb{R}^4$ to S^4 in instanton theory. (Technically, S^2 is the 'one-point compactification' (see Sims 1976) of the open two-dimensional disc.) Thus, the flux becomes quantised, for the same reason that the magnetic flux emanating from a Dirac monopole is quantised. The flux number can be interpreted, via the theory of characteristic classes, as a 'classification number' of a circle bundle over S^2. (See Bais (1982) for this approach.)

A second and rather more meaningful interpretation can be obtained if more physical information is introduced. (See Rajaraman (1982) for a more detailed discussion.) The Landau–Ginzburg model involves a complex scalar field, the order parameter, which, in order to maintain a finite free energy, must take on non-zero values on the 'circle at infinity' (in the two-dimensional case). Thus, the order parameter defines a map from S^1 to sets of complex numbers of constant modulus. Such maps are classified in the usual way by 'winding numbers', and the winding number can be related to the flux through the core of the specimen.

Of course, neither of these ways of considering superconductive flux quantisation is complete, in the sense that neither can yield the explicit condition

$$f = n(hc/2e)$$

(where f is the flux through the core, n is integral, h is Planck's constant, c is the speed of light, and e is the electron charge). This relation can only be obtained with the aid of quantum mechanical considerations; that is, one must consider the Schrödinger equation for a Cooper pair in the specimen (see, for example, Sakurai 1967). By requiring the wavefunction to be single valued, one obtains both flux quantisation and the magnitude of the flux quantum, $hc/2e$.

As Trautman (1970) has pointed out, the wavefunction has an extremely natural interpretation as a cross-section of an associated bundle of the electromagnetic gauge bundle. This fact, combined with the above remarks, leads us to ask whether it is possible to obtain a geometric or topological interpretation of superconductive flux quantisation by *directly* converting the usual quantum mechanical argument into geometric language. The immediate objective of this work is to prove that this can indeed be done. The method involves the important geometric concept of the holonomy group, but is completely independent of the approaches sketched earlier. (In particular, all principal bundles employed are trivial, so that there is no question of characteristic classes being employed.) It is hoped that this technique may yield some clues as to how the general theory of topological charge may be combined with quantum mechanics.

2. Geometric preliminaries

We adopt the usual geometric interpretation of gauge fields as connections on principal fibre bundles. (Trautman 1970, Bleecker 1981).

The existence of a connection Γ on a principal fibre bundle (P, M, G) (where M is the base manifold, G is the structural group, and $\pi: P \rightarrow M$ is the projection) allows us to associate a mapping from $\pi^{-1}(x)$ to $\pi^{-1}(y)$ with each curve γ joining $x \in M$ to $y \in M$. This is referred to as parallel transport (Kobayashi and Nomizu 1963). Parallel transport defines a natural equivalence relation (denoted $\sim$) on P: specifically, if $p, q \in P$, then $p \sim q$ if p and q can be joined by a horizontal curve in P. The equivalence class $Q(p)$ associated with a given $p \in P$ is called the *holonomy bundle* through p, while the subgroup of G defined by $\Phi(p) = \{g \in G \text{ such that } pg \sim p\}$ is called the *holonomy group* at p. Clearly p can only be parallel transported to pg along a closed curve in M. If we consider only null-homotopic curves in M, then we obtain the restricted holonomy group $\Phi^0(p)$.

It may be shown that, if M is connected and paracompact (as we shall assume henceforth), then

(i) The $\Phi(p)$ (for all $p \in P$) are mutually isomorphic Lie groups.

(ii) $\Phi^0(p)$ is a connected normal Lie subgroup of $\Phi(p)$, and there is a natural homomorphism from the first homotopy group of M to $\Phi(p)/\Phi^0(p)$.

(iii) Each triple $(Q(p), M, \Phi(p))$ is a reduced sub-bundle of (P, M, G).

(iv) The Lie algebra of $\Phi(p)$ is spanned by the restriction of the curvature form Ω to $Q(p)$.

We now consider parallel transport in associated bundles. Let (E, P, M, G, F) be an associated bundle of (P, M, G) with standard fibre F. Let $pf \in E$ (where pf denotes the equivalence class represented by $(p, f) \in P \times F$) and let γ be a closed curve in M with $\gamma(0) = \pi(p)$. It is easily shown that if g is the element of $\Phi(p)$ corresponding to γ, then the parallel transport (in E, induced by the connection in (P, M, G)) of pf along γ maps pf to pgf.

Let ϕ be a cross-section of E. We shall now assume that (P, M, G) is trivial. Let $\sigma: M \to P$ be a fixed global cross-section of P; then we can represent ϕ by a map $\psi: M \to F$ by setting $\phi(x) = \sigma(x)\psi(x)$ for each $x \in M$. If g is the holonomy element corresponding to γ, as above, then parallel transport will map $\sigma(x)\psi(x)$ to $\sigma(x)g\psi(x)$, and we may regard this as a mapping which replaces $\psi(x)$ by $g\psi(x)$. In this way we define the concept of parallel transport of ψ around γ.

3. The geometry of flux quantisation

Consider an annular specimen of superconductive material immersed in a static magnetic field parallel to its axis. As a result of the Meissner effect (Rose-Innes and Rhoderick 1978) the magnetic field inside the surface penetration zone is zero. In the static case, the electric field must also be zero, so that the electromagnetic field F is zero throughout the annular region, denoted M.

Since we are dealing with electromagnetism, we wish to consider U(1) bundles over M. Since M is topologically equivalent to S^1 (the circle), any principal U(1) bundle over M is necessarily trivial (Steenrod 1951); the bundle space P may be visualised as the ordinary two-dimensional torus T^2. As we shall see below, the triviality of this bundle is necessary for physical reasons also.

The equation $F = 0$ implies that the connection corresponding to the electromagnetic potentials is flat ($\Omega = 0$). However, this does not necessarily imply that parallel transport reverts to the identity map. Property (iv) listed above implies that the Lie algebra of the holonomy group is zero, but this only allows us to conclude that the connected component of the identity of $\Phi(p)$ is $\{e\}$ (where e is the identity). By property (ii), this connected component is precisely the restricted holonomy group $\Phi^0(p)$. In general, then, only $\Phi^0(p)$ need be trivial; from (ii), we see that $\Phi(p)$ will be a discrete group homomorphic (though not necessarily isomorphic) to the first homotopy group of the annulus M. The latter is of course just the additive group of integers, $\mathbb{Z}$.

None of this is directly relevant to the physics of the superconductor. It becomes relevant only when we introduce the Schrödinger equation. For our purposes, it is sufficient to treat the Cooper pairs as particles of charge $2e$ governed by the Schrödinger equation

$$(1/2m)[-(\mathrm{i}h/2\pi)\nabla - 2eA/c)^2\psi + V\psi = E\psi, \tag{1}$$

where the notation is standard. Since the magnetic field is zero in M, the solution takes the form (Sakurai 1967)

$$\psi(x) = \psi_0(x) \exp\left(\frac{4\pi \mathrm{i} e}{hc} \int^x A \cdot \mathrm{d}s\right), \tag{2}$$

where the line integral is taken along any curve terminating at x, and where ψ_0 is the solution if $A = 0$.

As remarked above, the underlying principal bundle here is the trivial bundle $(T^2, S^1, \mathrm{U}(1))$. Since U(1) has a natural action on the complex numbers C, we can construct the associated bundle with standard fibre C. Any cross-section of this associated bundle gives rise, in the way described in the preceding section, to a complex-valued function on M which (since it has the correct gauge transformation behaviour) may be identified with the wavefunction (Trautman 1970). The concept of parallel transport of the wavefunction around an arbitrary closed curve in M can therefore be defined as in the preceding section. Examining equation (2) above, we see that indeed ψ is obtained from ψ_0 precisely in this way. It is to be emphasised that this geometric relationship between ψ and ψ_0 arises not from the mathematical formulation *but rather from the physical conditions expressed by the Schrödinger equation.* There is no reason to expect *a priori* that ψ should be related to ψ_0 by parallel transport.

Let the integral in (2) be taken along a closed curve which winds around the core of the annulus once, and set $f = \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s}$. Then f is a constant and

$$\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) \exp(4\pi \mathrm{i} ef/hc). \tag{3}$$

However, for a curve with winding number N, f must be replaced by Nf, and so ψ will not be single-valued in general. This is a direct consequence of the fact, discussed above, that the condition of flatness is not sufficient to force the holonomy group to be equal to $\{e\}$. The requirement that ψ be single-valued can only be satisfied if we directly 'trivialise' the holonomy group by setting $f = n(hc/2e)$, where n is an integer. This is, of course, just the usual flux quantisation condition. The latter thus has a direct geometric interpretation as *the necessary and sufficient condition for a flat connection on* $(T^2, S^1, \mathrm{U}(1))$ *to have a trivial holonomy group.*

Notice that the triviality of the gauge bundle is essential throughout this discussion. This can be seen either with the aid of characteristic class theory or simply by noting that according to property (iii) listed in § 2, the holonomy bundles define global cross-sections if $\Phi(p) = \{e\}$.

We conclude with a discussion of the interpretation of the integer which occurs in the equation for f, that is, the 'flux number'. Let Γ be a flat connection with holonomy group equal to $\{e\}$ in a trivial bundle (P, M, G). Let Q be any holonomy bundle of Γ; by property (iii) of section 2, Q is a sub-bundle of (P, M, G) which has $\{e\}$ as structural group. Hence for each $x \in M$ we may define $Q(x) \in P$ as the unique element of $Q \cap \pi^{-1}(x)$. This defines a map $Q: M \to P$ with $\pi(Q(x)) = x$, or, in effect (since P is trivial) a map $\bar{Q}: M \to G$. It is easily shown that any other holonomy bundle defines a map homotopic to $\bar{Q}$, and that, in fact, there is a one-to-one correspondence between 'globally flat' (holonomy group $= \{e\}$) connections in a trivial bundle (P, M, G) and homotopy classes of maps from M to G. In the case of the bundle $(T^2, S^1, \mathrm{U}(1))$, this means that the different globally flat connections are classified by the winding numbers which describe the possible mappings from S^1 to $\mathrm{U}(1)$. This winding number is just the flux number n. Thus we see that the flux number may be interpreted geometrically as a classification number characterising the distinct globally flat connections on $(T^2, S^1, \mathrm{U}(1))$. This situation can be visualised by considering the surface of a cylinder, marked with lines parallel to its length extending from end to end. If one end is 'twisted' through an angle $n2\pi$ with respect to the other, and the ends are joined, then we will obtain a set of closed paths on the surface of the resulting torus. The torus is the bundle space of the above principal bundle, while the tangent vectors of the closed paths may be identified with the horizontal subspaces corresponding to a globally flat connection. The flux number corresponds to the number of 'twists' in the surface of the torus.

4. Conclusion

In this paper we have proposed a geometric interpretation of superconductive flux quantisation which is based directly on the quantum mechanical derivation of this effect. No use has been made of the theory of characteristic classes or of the behaviour of the Landau–Ginzburg order parameter. The flux quantisation effect itself is interpreted in terms of a *global flatness condition* (to be contrasted with local flatness, which corresponds to the Meissner effect), while the flux number is related to the classification of globally flat connections in a trivial bundle.

Apart from its intrinsic interest, this approach to the mathematical interpretation of superconductive flux quantisation may shed some light on the general theory of 'topological quantum numbers'. It is to be stressed that, from a mathematical point of view, the superconductive 'flux number' has almost nothing in common with the integer describing the number of flux units emanating from a Dirac magnetic monopole. (See Goddard and Olive 1978.) The latter classifies the various inequivalent U(1) bundles over S^2 in a way which is *completely independent* of any particular choice of connection. (This is the content of the Weil lemma—see Bleecker (1981).) By contrast, in the case of superconductivity, only one U(1) bundle can be constructed over the annulus, and the integer n classifies inequivalent *connections* instead of inequivalent *bundles*. Again, while it is true that both the superconductive and the monopole flux quantisation conditions can be derived from the requirement that the wavefunction be single-valued, this resemblance is essentially superficial. We have seen that in the former case, the necessity of enforcing single-valuedness arises from the fact that, in general, the local flatness (curvature = 0) of a connection does not imply global flatness (holonomy group = $\{e\}$). In the monopole case, however, it arises simply from the fact that a 'monopole bundle' is necessarily non-trivial (Wu and Yang 1975). (On the other hand, it is possible that there may be some kind of 'isomorphism' between the classification of bundles over S^2 and that of connections on the torus bundle over S^1. This possibility remains to be investigated.)

We conclude, therefore, that the superconductive topological quantum number has a mathematical interpretation which distinguishes it from 'monopole numbers' and 'instanton numbers' in a fundamental way.

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